VARIATIONAL ANALYSIS: AN OVERVIEW

BORIS MORDUKHOVICH

Wayne State University, USA

Lecture 1 at the **Forum on Developments and Origins of Operations Research**

Shenzhen, China, November 2021

ORGANIZERS: **Operations Research Society of China & Southern University of Science and Technology**

THE FUNDAMENTAL VARIATIONAL PRINCIPLE

Namely, because the shape of the whole universe is the most perfect and, in fact, designed by the wisest creator, nothing in all the world will occur in which no maximum or minimum rule is somehow shining forth...

Leonhard Euler (1744)

INTRINSIC NONSMOOTHNESS

is typically encountered in applications of modern variational principles and techniques to numerous problems arising in pure and applied mathematics particularly in analysis, geometry, dynamical systems (ODE,PDE), optimization, equilibrium, mechanics, control, economics, ecology, biology, computers science...

REMARKABLE CLASSES OF NONSMOOTH FUNCTIONS

MARGINAL/VALUE FUNCTIONS

$$\mu(x) := \inf \left\{ \varphi(x,y) \,\middle|\, y \in G(x) \right\}$$

crucial in perturbation and sensitivity analysis, stability, and many other issues. In particular, **DISTANCE FUNCTIONS**

$\text{dist}(x;\Omega) := \inf \left\{ \|x-y\| \,\middle|\, y \in \Omega \right\}$ or generally $\rho(x,z) := \text{dist}(x;F(z))$

naturally appear via variational principles and penalization

INTRINSIC NONSMOOTHNESS (cont.)

MAXIMUM FUNCTIONS

$$f(x) = \max_{u \in U} g(x, u),$$

in particular, **HAMILTONIANS** in physics, mechanics, calculus of variations, systems control, variational inequalities, etc.

NONSMOOTH/NONCONVEX SETS AND MAPPINGS

Parametric sets of feasible and optimal solutions in various problems of equilibrium, optimization, dynamics
Preference and production sets in economic modeling
Reachable sets in dynamical and control systems
Sets of Equilibria and **Equilibrium Constraints** in physical, mechanical, economic, ecological, and biological models

SUBDIFFERENTIALS

of $\varphi\colon I\!R^n \to \overline{I\!R} := (-\infty, \infty]$ with $\varphi(\bar{x}) < \infty$ should satisfy:

1) for convex functions φ reduces to

$$\partial^{\bullet}\varphi(\bar{x}) = \Big\{ v \Big| \; \varphi(x) - \varphi(\bar{x}) \geq \langle v, x - \bar{x} \rangle \;\; \text{for all} \;\; x \in I\!R^n \Big\}$$

2) If $\bar{x}$ is a local minimizer for φ, then $0 \in \partial^{\bullet}\varphi(\bar{x})$

3) Sum Rule (Basic Calculus)

$$\partial^{\bullet}(\varphi_1 + \varphi_2)(\bar{x}) \subset \partial^{\bullet}\varphi_1(\bar{x}) + \partial^{\bullet}\varphi_2(\bar{x})$$

4) Robustness

$$\partial^{\bullet}\varphi(\bar{x}) = \Big\{ v \in I\!R^n \Big| \; \exists\, x_k \to \bar{x}, \; v_k \to v, \; \varphi(x_k) \to \varphi(\bar{x}), \;\; \text{s.t.} \;\; v_k \in \partial^{\bullet}\varphi(x_k), \quad k = 1, 2, \ldots \Big\}$$

THE BASIC SUBDIFFERENTIAL

of $\varphi\colon I\!R^n \to \overline{I\!R}$ at $\bar{x}$ is defined by the outer limit

$$\partial\varphi(\bar{x}) = \Big\{ v \in I\!R^n \Big|\ \exists\, x_k \to \bar{x},\ v_k \to v,\ \varphi(x_k) \to \varphi(\bar{x}), \text{ s.t. } \ v_k \in \widehat{\partial}\varphi(x_k), \quad k = 1, 2, \ldots \Big\}$$

where regular/viscosity subdifferential of φ at x is defined by

$$\widehat{\partial}\varphi(x) := \left\{ v \,\Big|\, \liminf_{u \to x} \frac{\varphi(u) - \varphi(x) - \langle v, u - x \rangle}{\|u - x\|} \geq 0 \right\}$$

The **basic subdifferential** is **minimal** among all subdifferentials satisfying 1)-4), nonempty

$$\partial\varphi(\bar{x}) \neq \emptyset \ \text{ for Lipschitz functions}$$

while often nonconvex, e.g., $\partial(-|x|)(0) = \{-1, 1\}$. Moreover, its convexification, made for convenience, can dramatically worsen the basic properties and applications

VARIATIONAL GEOMETRY

The (basic, limiting) **NORMAL CONE** $N(\bar{x};\Omega) := \partial\delta(\bar{x};\Omega)$ to Ω at $\bar{x} \in \Omega$, where $\delta(x;\Omega) := 0$ for $x \in \Omega$ and $\delta(x;\Omega) := \infty$ for $x \notin \Omega$. It can be equivalently described as

$$N(\bar{x};\Omega) = \Big\{ v \in I\!R^n \Big| \ \exists\, x_k \to \bar{x},\ \alpha_k \geq 0,\ w_k \in \Pi(x_k;\Omega) \\ \text{s.t.}\quad \alpha_k(x_k - v_k) \to v \ \text{ as } \ k \to \infty \Big\},$$

where $\Pi(x;\Omega)$ is the Euclidean projector of the set Ω, i.e.,

$$\Pi(x;\Omega) := \Big\{ y \in \Omega \Big| \ \|y - x\| = \min_{u \in \Omega} \|u - x\| \Big\}$$

EXTREMALITY OF SET SYSTEMS

DEFINITION. A vector $\bar{x} \in \Omega_1 \cap \Omega_2$ is a **LOCAL EXTREMAL POINT** of the system of closed sets $\{\Omega_1, \Omega_2\}$ in $I\!R^n$ if there exists a neighborhood U of $\bar{x}$ such that for any $\varepsilon > 0$ there is $a \in I\!R^n$ with $\|a\| < \varepsilon$ satisfying

$$(\Omega_1 + a) \cap \Omega_2 \cap U = \emptyset$$

EXAMPLES:

—boundary point of closed sets

—local solutions to constrained optimization, multiobjective optimization, and other optimization-related problems

—minimax solutions and equilibrium points

—Pareto-type allocations in economics

—stationary points in mechanical and ecological models, etc.

EXTREMAL PRINCIPLE

THEOREM. Let $\bar{x}$ be a **LOCAL EXTREMAL POINT** for the system of closed sets $\{\Omega_1, \Omega_2\}$ in $I\!R^n$.Then there exists a dual element $0 \neq v \in I\!R^n$ such that

$$v \in N(\bar{x}; \Omega_1) \cap (-N(\bar{x}; \Omega_2))$$

This is a **VARIATIONAL** counterpart of the separation theorem for the case of nonconvex sets, which plays a fundamental role in variational analysis and its applications.
SOME APPLICATIONS: Full Calculus of Generalized Differentiation; Well-Posedness in Optimization and Optimality Conditions; Sensitivity Analysis, ODE and PDE Control, Economic and Mechanical Equilibria, Numerical Analysis...

MATHEMATICAL PROGRAMMING

Consider the nonsmooth NP problem:

$$\text{minimize } \varphi_0(x) \text{ subject to } \begin{array}{l} \varphi_i(x) \le 0,\ i = 1, \ldots, m \\ \varphi_i(x) = 0,\ i = m+1, \ldots, m+r \\ x \in \Omega \end{array}$$

THEOREM (generalized Lagrange multipliers). Let φ_i be **locally Lipschitzian** and Ω be locally closed around an optimal solution $\bar{x}$. Then there are $(\lambda_0, \ldots, \lambda_{m+r}) \neq 0$ satisfying

$$\lambda_i \ge 0,\ i = 0, \ldots, m, \quad \lambda_i \varphi_i(\bar{x}) = 0,\ i = 1, \ldots, m,$$
$$0 \in \partial\Big(\sum_{i=0}^{m+r} \lambda_i \varphi_i\Big)(\bar{x}) + N(\bar{x}; \Omega)$$

Moreover, $\lambda_0 \neq 0$ **(Normality)** under appropriate **Constraint Qualification Conditions**

DYNAMICAL SYSTEMS

governed by differential inclusions

$$\dot{x}(t) \in F(x(t), t), \quad t \in [a, b], \quad x(a) = x_0 \in I\!R^n$$

where $\dot{x}$ stands for the time derivative and where $F\colon I\!R^n \times [a, b] \rightrightarrows I\!R^n$ is a set-valued mapping. This description is important for qualitative theory of dynamical system and numerous applications, e.g., to various economic, ecological, biological, financial systems, climate research...

In particular, this covers parameterized control systems with

$$\dot{x} = g(x, u, t), \;\; u(\cdot) \in U(x, t)$$

where the control region $U(x, t)$ depends on time and state

DISCRETE APPROXIMATIONS

Euler's finite difference (for simplicity)

$$\dot{x}(t) \approx \frac{x(t+h) - x(t)}{h}, \;\; h \to 0$$

Consider the mesh as $N \to \infty$

$t_j := a + jh_N, \;\; j = 0, \ldots, N, \quad t_0 = a, \; t_N = b, \;\; h_N = (b-a)/N$

Discrete Inclusions

$$x_N(t_{j+1}) \in x_N(t_j) + h_N F(x_N(t_j), t_j)$$

with piecewise linear Euler broken lines

Various Well-Posedness, Convergence, and Stability Issues of Numerical and Qualitative Analysis in Finite-Dimensional and Infinite-Dimensional Spaces

DISCRETE APPROXIMATIONS

Euler's finite difference (for simplicity)

$$\dot{x}(t) \approx \frac{x(t+h) - x(t)}{h}, \ \ h \to 0$$

Consider the mesh as $N \to \infty$

$t_j := a + jh_N, \ \ j = 0, \ldots, N, \quad t_0 = a, \ t_N = b, \ \ h_N = (b-a)/N$

Discrete Inclusions

$$x_N(t_{j+1}) \in x_N(t_j) + h_N F(x_N(t_j), t_j)$$

with piecewise linear Euler broken lines

Various Well-Posedness, Convergence, and Stability Issues of Numerical and Qualitative Analysis in Finite-Dimensional and Infinite-Dimensional Spaces

OPTIMAL CONTROL OF DIFFERENTIAL INCLUSIONS

minimize the cost functional

$$J[x] = \varphi\big(x(b)\big) \quad \text{subject to}$$

$$\dot{x}(t) \in F(x(t), t) \quad \text{a.e.} \quad t \in [a, b], \qquad x(a) = x_0$$

$$x(b) \in \Omega \subset I\!R^n$$

where $F\colon I\!R^n \rightrightarrows I\!R^n$ is a Lipschitz continuous set-valued mapping, Ω is a closed set, φ is a l.s.c. function
This covers various open-loop and closed-loop control systems with ODE dynamics and hard control and state constraints

CODERIVATIVES OF MAPPINGS

Let $F\colon X \rightrightarrows Y$ be a set-valued mapping with $(\bar{x},\bar{y}) \in \text{gph}F$. Then $D^*F(\bar{x},\bar{y})\colon Y^* \rightrightarrows X^*$ defined by

$$D^*F(\bar{x},\bar{y})(y^*) := \left\{x^* \middle| \, (x^*,-y^*) \in N((\bar{x},\bar{y}); \text{gph}\, F)\right\}$$

is called the **coderivative** of F at $(\bar{x},\bar{y})$.

If $F\colon X \to Y$ is smooth around $\bar{x}$, then

$$D^*F(\bar{x})(y^*) = \left\{\nabla F(\bar{x})^* y^*\right\} \quad \text{for all} \;\; y^* \in Y^*$$

i.e., the coderivative is a proper generalization of the classical adjoint derivative. If $F\colon X \to Y$ is single-valued and locally Lipschitzian around $\bar{x}$, then the scalarization formula holds:

$$D^*F(\bar{x})(y^*) = \partial\langle y^*, F\rangle(\bar{x})$$

ENJOY FULL CALCULUS!

CHARACTERIZATION OF WELL-POSEDNESS

$F\colon X \rightrightarrows Y$ is called Lipschitz-like around $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if there are neighborhood U of $\bar{x}$, V of $\bar{y}$ and modulus $\ell \geq 0$ for which we have the inclusion

$$F(x) \cap V \subset F(u) + \ell\|x-u\|I\!B \quad \text{for all} \quad x,u \in U$$

THEOREM. F is Lipschitz-like around $(\bar{x},\bar{y})$ if and only if

$$D^*F(\bar{x},\bar{y})(0) = \{0\}$$

The exact bound of Lipschitz moduli ℓ is calculated by

$$\operatorname{lip} F(\bar{x},\bar{y}) = \|D^*F(\bar{x},\bar{y})\|$$

EXTENDED EULER-LAGRANGE+MAXIMUM PRINCIPLE

THEOREM. Let $\bar{x}(\cdot)$ be an optimal solution to the control problem. Then one has
Euler-Lagrange inclusion

$$\dot{p}(t) \in \mathrm{co}\, D^*F\Big(\bar{x}(t), \dot{\bar{x}}(t)\Big)\Big(-p(t)\Big) \;\text{ a.e.,}$$

Weierstrass-Pontryagin maximum condition condition

$$\Big\langle p(t), \dot{\bar{x}}(t)\Big\rangle = \max_{v \in F(\bar{x}(t))} \Big\langle p(t), v\Big\rangle \;\text{ a.e.}$$

transversality condition

$$-p(b) \in \lambda \partial \varphi\Big(\bar{x}(b)\Big) + N\Big(\bar{x}(b); \Omega\Big)$$

with nontriviality condition $\Big(\lambda, p(\cdot)\Big) \neq 0$.
PROOF: DISCRETE APPROXIMATIONS

HAMILTONIAN CONDITION

THEOREM. Let the sets $F(x) \subset \mathbb{R}^n$ be convex. Then the extended Euler-Lagrange inclusion is equivalent to the extended Hamiltonian inclusion

$$\dot{p}(t) \in \text{co}\left\{u \middle| \Big(-u, \dot{x}(t)\Big) \in \partial H\Big(\bar{x}(t), p(t)\Big)\right\} \text{ a.e.}$$

in terms of the basic subdifferential of the (true) Hamiltonian

$$H(x,p,t) := \sup\left\{\Big\langle p, v\Big\rangle \middle| v \in F(x,t)\right\}$$

which is intrinsically nonsmooth

SEMILINEAR EVOLUTION INCLUSIONS AND PDEs

minimize $J[x] := \varphi(x(b))$ subject to
mild solutions to the semilinear evolution inclusion

$$\dot{x}(t) \in Ax(t) + F(x(t), t), \quad x(a) = x_0$$

with the endpoint constraints

$$x(b) \in \Omega \subset X$$

where A is an unbounded generator of the C_0 semigroup, i.e.

$$\begin{aligned} &x(t) = e^{A(t-a)}x_0 + \int_a^t e^{A(t-s)}v(s)\, ds, \quad t \in [a, b] \\ &v(t) \in F(x(t), t), \quad t \in [a, b] \end{aligned}$$

in the sense of Bochner integration.
Cover PDE systems with parabolic and hyperbolic dynamics

CONTROLLED SWEEPING PROCESS

described by: minimize

$$J[x] := \varphi(x(T)) + \int_0^T f(x(t), \dot{x}(t))\, dt \text{ s.t.}$$

$$\dot{x}(t) \in -N(x(t); C(t)), \quad t \in [0, T], \quad x(0) = x_0$$

with the controlled moving set

$$C(t) = \{x \in I\!R^n |\ \langle u(t), x \rangle \le b(t)\}, \quad \|u(t)\| = 1$$

Shape optimization governed by discontinuous differential inclusions. The method of discrete approximations does the job

REFERENCES

• **R. T. ROCKAFELLAR and R. J-B. WETS**, **VARIATIONAL ANALYSIS**, Grundlehren Series (Fundamental Principles of Mathematical Sciences), Vol. **317**, Springer, 1998.

• **B. S. MORDUKHOVICH**, **VARIATIONAL ANALYSIS AND GENERALIZED DIFFERENTIATION, I: BASIC THEORY, II: APPLICATIONS**, Grundlehren Series (Fundamental Principles of Mathematical Sciences), Vol. **330, 331**, Springer, 2006. Chinese translation in Science Press, Beijing, 2011, 2014.

• **B. S. MORDUKHOVICH**, **VARIATIONAL ANALYSIS AND APPLICATIONS**, Series: Monographs in Mathematics, Springer, 2018. Chinese translation in Science Press, Beijing, 2022.

- **B. S. MORDUKHOVICH and N. M. NAM**, **CONVEX ANALYSIS AND BEYOND, I: BASIC THEORY**, Series: Operations Research and Financial Engineering, Springer, 2022.