AN OVERVIEW OF VARIATIONAL ANALYSIS

5. SOLUTION MAPPINGS AND STABILITY

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Classical Perspective on "Problems" and "Solutions"

Problems modeled by equations — here finite-dimensional

for differentiable $F : \boldsymbol{R}^n \to \boldsymbol{R}^n$, find $\bar{x}$ such that $F(\bar{x}) = 0$

optimization? $\supset$ the case where F is a gradient mapping

Error analysis: issues important to computation

when $F(\bar{x}) = 0$ but a known x has $F(x) = v \neq 0$, can the distance of x from $\bar{x}$ be estimated from size of v?

Solution mappings — capturing dependence on parameters

for $F : \boldsymbol{R}^d \times \boldsymbol{R}^n \to \boldsymbol{R}^n$, let $S(p) = \{x \mid F(p,x) = 0\}$

- S is set-valued, but may have a "single-valued localization"
- the workhorse there is the standard implicit function theorem
- error analysis $\longleftrightarrow$ the case of $S(v) = \{x \mid F(x) - v = 0\}$

New Perspective That Welcomes Set-Valuedness

aimed at solving optimality conditions and much more

Problems modeled by "inequations"

for set-valued $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, find $\bar{x}$ such that $M(\bar{x}) \ni 0$, i.e.,
$\bar{x} \in M^{-1}(0)$ for the inverse mapping $M^{-1} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$

optimization? $\supset$ the case where M is a subgradient mapping

Error analysis: if x has $M(x) \ni v \neq 0$, can the size of v yield an estimate of the distance of x from the solution set $M^{-1}(0)$?

Solution mappings at this level of generality

for $M : \mathbb{R}^d \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, let $S(p) = \{x \mid M(p,x) \ni 0\}$

- how "stable" is a solution $\bar{x} \in S(\bar{p})$ when $\bar{p}$ shifts to p?
- can a version of the implicit function theorem provide help?

Direct Examples in Optimization

Solving basic subgradient conditions

for lsc proper f on $\mathbf{R}^n$ find $\bar{x}$ such that $0 \in \partial f(\bar{x})$

aimed at identifying a local minimum of f

Associated "basic" solution mapping: $S(v) = \{ x \mid v \in \partial f(x) \}$

$=$ the solution set to condition for minimizing $f(x) - v{\cdot}x$ in x

v = "**tilt**" perturbation, to be studied around $v = 0$

Solving optimality conditions involving dual variables

for f on $\mathbf{R}^n \times \mathbf{R}^m$ find $\bar{x}$, $\bar{y}$, such that $(0, \bar{y}) \in \partial f(\bar{x}, 0)$

aimed at local solutions to minimizing $f(x, u)$ subject to $u = 0$

Associated "primal-dual" solution mapping:

$S(v, u) = \{ (x, y) \mid (v, y) \in \partial f(x, u) \}$ around $(v, u) = (0, 0)$

A Key Problem Model in This Set-Valued Extension

Solving a "generalized equation"

find $\bar{x}$ such that $M(\bar{x}) \ni 0$ in the case of $M(x) = F(x) + N(x)$

equivalently, find $\bar{x}$ such that $-F(\bar{x}) \in N(\bar{x})$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{C}^1$ and $N : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ has closed graph

Parametric version: $F(p, x)$, but N kept fixed

solution mapping: $S(p) = \{\, x \mid -F(p, x) \in N(x) \,\}$

Immediate examples: of what this formulations covers

- solve $F(\bar{x}) = 0$ (*classical equation*): where $N(x) \equiv \{0\}$
- solve $-\nabla f_0(\bar{x}) \in N_C(\bar{x})$ (*optimization*): $G = \nabla f_0$, $N = N_C$

$N_C(x) =$ normal cone if $x \in C$, but $N_C(x) = \emptyset$ if $x \notin C$

The "Variational Inequality" Case

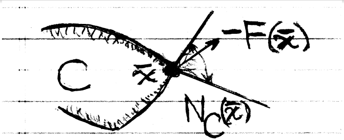
Variational inequality problem — geometric type

solve $-F(\bar{x}) \in N_C(\bar{x})$ in the case of a closed convex set C

Why "inequality"? because for a convex set C,

$v \in N_C(\bar{x}) \iff \bar{x} \in C$ and $v \cdot (x - \bar{x}) \le 0$ for all $x \in C$

$-F(\bar{x}) \in N_C(\bar{x}) \iff \bar{x} \in C$ and $F(\bar{x}) \cdot (x - \bar{x}) \ge 0, \forall x \in C$



Variational inequality problem — general type

solve $-F(\bar{x}) \in \partial\varphi(\bar{x})$ in the case of φ lsc proper convex, i.e.,

find $\bar{x}$ such that $\varphi(x) \ge \varphi(\bar{x}) - F(\bar{x}) \cdot (x - \bar{x})$ for all x

the "geometric" type has $\varphi = \delta_C$, so that $\partial\varphi = N_C$

Optimization V.I. Example With Lagrangian Format

Constraints in local optimality: specializing $-\nabla f_0(\bar{x}) \in N_C(\bar{x})$

minimize $f_0(x)$ over $C = \{x \mid (f_1(x), \dots, f_m(x)) \in D\} \cap X$
for $\mathcal{C}^1$ functions f_i, a convex set X, and a convex cone D

Normal cone formula: via earlier calculus, under constraint qual.

$-\nabla f_0(\bar{x}) \in N_C(\bar{x}) \iff \exists\, \bar{y} \in N_D(\bar{u})$ for $\bar{u} = (f_1(\bar{x}), \dots, f_m(\bar{x}))$
such that $-\nabla f_0(\bar{x}) \in N_X(\bar{x}) + \bar{y}_1 \nabla f_1(\bar{x}) + \cdots + \bar{y}_m \nabla f_m(\bar{x})$,
where $\bar{y} \in N_D(\bar{u}) \iff \bar{u} \in N_Y(\bar{y})$ for the polar cone $Y = D^*$

Lagrangian translation — with $L(x, y) = f_0(x) + \sum_{i=1}^m y_i f_i(x)$

$-\nabla_x L(\bar{x}, \bar{y}) \in N_X(\bar{x}), \;\; \nabla_y L(\bar{x}, \bar{y}) \in N_Y(\bar{y})$, or equivalently,
$-F(x, y) \in N_{X \times Y}(\bar{x}, \bar{y})$ for $F(x, y) = (\nabla_x L(x, y), -\nabla_y L(x, y))$
a variational inequality model of geometric type

Parameterization: functions $f_i(x, p)$, but X and D kept fixed
$-\nabla f_0(\bar{x}) \in N_{C(p)}(\bar{x})$ is reduced to $-F(p, x, y) \in N_{X \times Y}(x, y)$

Implicit and Inverse Function Theorems — Background

Equation problem: when $F(\bar{p}, \bar{x}) = 0$ for $F : \boldsymbol{R}^d \times \boldsymbol{R}^n \to \boldsymbol{R}^n$

determine if, $\forall\, p$ near $\bar{p}$, $\exists$ unique x near $\bar{x}$ solving $F(p, x) = 0$

$\longrightarrow$ **implicit** function s such that $F(p, s(p)) = 0$

Inverse problem: when $G(\bar{x}) = \bar{v}$ for $G : \boldsymbol{R}^n \to \boldsymbol{R}^n$

determine if, $\forall\, v$ near $\bar{v}$, $\exists$ unique x near $\bar{x}$ solving $G(x) = v$

$\longrightarrow$ **inverse** function t such that $G(t(v)) = v$

observation: "inverse" $\subset$ "implicit" via $F(v, x) = G(x) - v$

Standard implicit function theorem — insightful formulation

The equation problem with F in $\mathcal{C}^1$ has YES answer if, for

$G(x) = F(\bar{p}, \bar{x}) + \nabla_x F(\bar{p}, \bar{x})(x - \bar{x})$ linearizing $F(\bar{p}, \cdot)$ at $\bar{x}$,

the inverse problem for G with $\bar{v} = 0$ has YES answer

moreover the implicit function s is then $\mathcal{C}^1$ etc.

Companion fact: the linearized inverse problem has YES answer

$\iff$ the Jacobian $\nabla_x F(\bar{p}, \bar{x})$ has full rank n

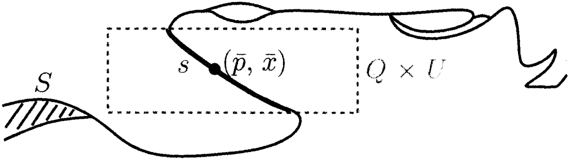
Localizations of Solution Mappings

on the way to extending from *equations* to *generalized equations*

$\bar{x} \in S(\bar{p})$ for a mapping $S : \boldsymbol{R}^d \rightrightarrows \boldsymbol{R}^n$ with closed graph

Single-valued localization of S around $(\bar{p}, \bar{x}) \in \operatorname{gph} S$

a single-valued mapping s obtained locally around $\bar{p}$ by taking
$\operatorname{gph} s = [Q \times U] \cap \operatorname{gph} S$ for a neighborhood $Q \times U$ of $(\bar{p}, \bar{x})$



Classical focus: differentiable s, formula for Jacobian $\nabla s(\bar{p})$

New focus: Lipschitz continous s, estimate of modulus $\operatorname{lip} s(\bar{p})$

$$\operatorname{lip} s(\bar{p}) := \inf \left\{ \lambda \,\middle|\, |s(p) - s(p')| \leq \lambda |p - p'| \text{ for } p, p', \text{ near } \bar{p} \right\}$$

differentiability not natural for optimization-oriented analysis

Implicit Function Theorem for *Generalized* Equations

Generalized equation framework: with F in $\mathcal{C}^1$, closed graph S

mapping $S(p) = \{x \mid -F(p,x) \in N(x)\}$ with $\bar{x} \in S(\bar{p})$

focus: a possible single-valued localization s with $s(\bar{p}) = \bar{x}$

Auxiliary inverse framework: G = linearization of $F(\bar{p},\cdot)$ at $\bar{x}$

mapping $T(v) = \{x \mid v - G(x) \in N(x)\}$, having $\bar{x} \in T(0)$

focus: a possible single-valued localization t with $t(0) = \bar{x}$

Robinson's Theorem — main version

S has a Lipschitz continuous single-valued localization s when
T has a Lipschitz continuous single-valued localization t,

and then $\operatorname{lip} s(\bar{p}) \leq \operatorname{lip} t(0) \cdot \|\nabla_p F(\bar{x},\bar{p})\|$

Left for particular situations: verifying the assumption on T

other versions of this result drop differentiability and
utilize other "approximations" of F than "linearization"

Lipschitz-like Properties of Set-Valued Mappings

Notation: B = closed unit ball,

$C + \varepsilon B$ = set of all x at distance $\leq \varepsilon$ from C

Aubin property: of $S : R^d \rightrightarrows R^n$ at $(\bar{p}, \bar{x}) \in \operatorname{gph} S$

$\exists$ neighborhoods Q of $\bar{p}$, U of $\bar{x}$, and $\lambda \geq 0$ such that

$$S(p) \cap U \subset S(p') + \lambda |p - p'| B \quad \text{for } p, p' \in Q$$

$\operatorname{lip} S(\bar{p}\,|\,\bar{x}) :=$ inf of λ for which this can hold locally

Calmness property: of $S : R^d \rightrightarrows R^n$ at $(\bar{p}, \bar{x}) \in \operatorname{gph} S$

$\exists$ neighborhoods Q of $\bar{p}$, U of $\bar{x}$, and $\lambda \geq 0$ such that

$$S(p) \cap U \subset S(\bar{p}) + \lambda |p - \bar{p}| B \quad \text{for } p \in Q$$

$\operatorname{clm} S(\bar{p}\,|\,\bar{x}) :=$ inf of λ for which this can hold locally

Corresponding implicit function theorems — same pattern!

S has the property if T has it, and then get estimate

Error Analysis in Terms of Metric Regularity

Orientation: with any $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ having closed graph

- interested in solving $M(\bar{x}) \ni 0$, **solution set** $= M^{-1}(0)$
- computations encounter x merely "close" to being a solution

solution error: $\mathrm{dist}(x, M^{-1}(0))$ **desired knowledge**

residual error: $\mathrm{dist}(0, M(x))$ **accessible knowledge**

Metric regularity — at a solution $\bar{x}$ in $M^{-1}(0)$

$\exists$ neighborhoods U of $\bar{x}$, V of 0, and $\kappa \geq 0$ such that

$$\mathrm{dist}(x, M^{-1}(v)) \leq \kappa\,\mathrm{dist}(v, M(x)) \text{ for } x \in U,\ v \in V$$

$\mathrm{reg}\, M(\bar{x}\,|\,0) :=$ inf of κ for which this can hold locally

Metric subregularity — at a solution $\bar{x}$ in $M^{-1}(0)$

$\exists$ neighborhood U of $\bar{x}$ and $\kappa \geq 0$ such that

$$\mathrm{dist}(x, M^{-1}(0)) \leq \kappa\,\mathrm{dist}(0, M(x)) \text{ for } x \in U$$

$\mathrm{subreg}\, M(\bar{x}\,|\,0) :=$ inf of κ for which this can hold locally

Connection with Inverse Mapping Theorems

Orientation: with $M : R^n \rightrightarrows R^m$ having closed graph

- interpret M^{-1} as the solution mapping S for $M(x) - v \ni 0$
- bring in the Lipschitz-like properties for such a mapping S

Inverse characterization of metric regularity

M is metrically regular at $\bar{x} \in M^{-1}(0)$ $\iff$
M^{-1} has the Aubin property at $(0, \bar{x}) \in \operatorname{gph} M^{-1}$,
and then $\operatorname{reg} M(\bar{x}\,|\,0) = 1/\operatorname{lip} M^{-1}(0\,|\,\bar{x})$

Inverse characterization of metric subregularity

M is metrically subregular at $\bar{x} \in M^{-1}(0)$ $\iff$
M^{-1} has the calmness property at $(0, \bar{x}) \in \operatorname{gph} M^{-1}$,
and then $\operatorname{subreg} M(\bar{x}\,|\,0) = 1/\operatorname{clm} M^{-1}(0\,|\,\bar{x})$

Consequence: in general equation-type cases where $M = F + N$, criteria can be derived from the extended implicit function theory

Tilt Stability in Local Optimality

Background problem: "locally" minimize lsc proper f on $\boldsymbol{R}^n$

candidates for local min: points $\bar{x}$ having $0 \in \partial f(\bar{x})$

approximate candidates: x having $v \in \partial f(x)$ for v near 0

Error analysis framework: with solution mapping $S : \boldsymbol{R}^n \rightrightarrows \boldsymbol{R}^n$

$$S(v) = \{x \mid v \in \partial f(x)\} = (\partial f)^{-1}(v), \text{ around } (0, \bar{x}) \in \operatorname{gph} S$$

$\longrightarrow$ look at metric regularity, relate to sufficiency for local min

Tilt stability: with respect to $\bar{x}$ being a <u>local</u> <u>minimizer</u> of f

$\exists$ neighborhoods V of 0 and U of $\bar{x}$ such that the mapping

$$V \ni v \mapsto \operatorname*{argmin}_{x \in U} \{ f(x) \underbrace{- v \cdot (x - \bar{x})}_{\text{tilting term}} \}$$ is <u>single-valued Lip. contin.</u>

Tilt stability versus metric regularity

tilt stability holds at $\bar{x}$ $\iff$ ∂f is metrically regular at $\bar{x}$ for 0

$\iff$ ∂f strongly monotone locally around $(\bar{x}, 0)$

Local Monotonicity of Subgradient Mappings

what is the strong local monotonicity that's tied to tilt stability?

Local monotonicity: of ∂f around $(\bar{x}, 0) \in \operatorname{gph} \partial f$

$\exists\, U \times V$ neighborhood* of $(\bar{x}, 0)$ such that

$$(x_1 - x_0)\cdot(v_1 - v_0) \geq 0 \quad \forall (x_0, v_0), (x_1, v_1) \in [U \times V] \cap \operatorname{gph} \partial f$$

strong: $(x_1 - x_0)\cdot(v_1 - v_0) \geq \sigma |x_1 - x_0|^2$, $\sigma > 0$

Recall global monotonicity — the case of $U \times V = R^n \times R^n$

monotonicity $\Longleftrightarrow$ convexity of f, and moreover

strong monotonicity $\Longleftrightarrow$ strong convexity of f

maybe this also extends to "local $\longleftrightarrow$ local"? **no!**

Variational convexity: a property of f that has been shown to characterize the local monotonicity of ∂f (and a strong version)

$\not\Longrightarrow$ f locally convex!

this is valuable in advancing the theory of local optimality

Further Study

- Lipschitzian properties of single-valued and set-valued mappings are treated in detail in Chapter 9 of Variational Analysis, where metric regularity also makes an appearance. However, the main source for results behing this lecture is the my later book with Asen Dontchev, Implicit Functions and Solution Mappings.

- For more on tilt stability and local monotonicity of subgradient mappings, which get into variational convexity, sources beyond those books will need to be consulted. The best way to begin is with articles #252 and #256 on my website. This topic is closely connected with ongoing research into second-order aspects of local optimality and how it can serve in guiding numerical methodology.

website: sites.math.washington.edu/∼rtr/mypage.html

And a New Book That Can Help

An Optimization Primer

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